Physics (H)- SEM-II_CC IV WAVES AND OPTICS_PB_Online Class-2

In our previous classes we have studied superposition of two harmonic oscillators with the same frequencies but of different amplitudes and phases. The result was again a harmonic oscillator of the same frequency, with its amplitude and phase determined by those of its components.

Now we will consider the superposition of two harmonic oscillators with different frequencies and discuss the phenomenon of production of beats

What are Beats:

When two waves of nearly equal frequencies travelling in a medium along the same direction superimpose upon each other, the amplitude of the resultant sound at a point rises and falls regularly. Consequently the intensity of the resultant sound at a point rises and falls regularly with time. This regular rise and fall of sound are called beats. When the sound intensity rises to maximum we call it as waxing of sound, when it falls to minimum we call it as waning of sound. Hence beats is the phenomenon of waxing and waning of sound due to interference of two sound waves of nearly equal frequencies. The number of beats produced per second is called beat frequency, which is equal to the difference in frequencies of two waves.

Analytical method:

Suppose that we have two collinear harmonic oscillations of different frequencies and amplitudes. For simplicity, we assume that the two oscillations have the same initial phase, which is taken to be zero to make the calculation a beat easier. The two SHMs with displacements $x_1$and x_2 and angular frequencies and ω_1 and ω_2 respectively, where $\omega_1 < \omega_2$, can be written as,

$$x_1 = A_1 \sin \omega_1 t$$

$$x_2 = A_2 \sin \omega_2 t$$

From the superposition principle, the resultant of these two oscillation is given by

$$x = x_1 + x_1 = A_1 \sin \omega_1 + A_2 \sin \omega_2$$

Now, to simplify the above equation, we define ω_a the average frequency and ω_m, the modulation frequency in the following way:

$$\omega_a = \frac{\omega_1+\omega_2}{2} \text{ and } \omega_m = \frac{\omega_2-\omega_1}{2}$$

$$\text{Hence, } \omega_1 = \omega_a - \omega_m \text{ and } \omega_2 = \omega_a + \omega_m$$

And now,

$$x = A_1 \sin(\omega_a - \omega_m) t + A_2 \sin(\omega_a + \omega_m) t$$

$$x = A_1(\sin\omega_a t\cos\omega_m t - \sin\omega_m t\cos\omega_a t) + A_2(\sin\omega_a t\cos\omega_m t + \sin\omega_m t\cos\omega_a t)$$

or,

$$x = [(A_1 + A_2)\cos\omega_m t]\sin\omega_a t - [(A_1 - A_2)\sin\omega_m t]\, cos\,\omega_a t$$

Let us now make following substitution,

$$[(A_1 + A_2)\cos\omega_m t] = A(t) cos\,\varepsilon(t)$$

And

$$-[(A_1 - A_2)\sin\omega_m t] = A(t) sin\,\varepsilon(t)$$

Then,

$$x = A(t)\sin\omega_a t\cos\varepsilon + A(t)\, cos\,\omega_a t\sin\varepsilon$$

Or,

$$x = A(t) sin(\omega_a t + \varepsilon)$$

Thus, the resultant oscillation obtained on superposition of two collinear simple harmonic oscillation with slightly different frequencies and amplitudes is again a simple harmonic oscillation with an amplitude (A) and phase-factor (ε) that is determined by the amplitude and frequencies of constituent oscillations and varies regularly with time.

The phase-factor of the resultant oscillation is given by,

$$\tan\varepsilon(t) = \frac{-(A_1 - A_2)\sin\omega_m t}{(A_1 + A_2)\cos\omega_m t}$$

But our centre of interest, in this discussion, is the variation of amplitude of resultant oscillation with time and thus appearance of beats.

The amplitude of resultant oscillation is given by,

$$A(t) = \sqrt{[(A_1 - A_2)\sin\omega_m t]^2 + [(A_1 + A_2)\cos\omega_m t]^2}$$

Now,

$$[(\boldsymbol{A_1} - \boldsymbol{A_2})\,\mathbf{sin}\,\boldsymbol{\omega_m}\,\boldsymbol{t}]^2 + [(\boldsymbol{A_1} + \boldsymbol{A_2})\,\mathbf{cos}\,\boldsymbol{\omega_m}\,\boldsymbol{t}]^2$$

$$(A_1^2 + A_2^2)(\sin^2\omega_m t + \cos^2\omega_m t) + 2A_1A_2(\cos\omega_m t - \sin\omega_m t)$$

$$= (A_1^2 + A_2^2) + 2A_1A_2(\cos^2\omega_m t - \sin^2\omega_m t)$$

$$= (A_1^2 + A_2^2) + 2A_1A_2\cos(2\omega_m t)$$

Therefore, $A(t) = \sqrt{{A_1}^2 + {A_2}^2 + 2A_1A_2\cos(2\,\omega_m t)}$

Condition for maximum amplitude:

Resultant amplitude assumes maximum value, $A(t) = (A_1 + A_2)$, whenever, $\cos(2\,\omega_m\,t) = +1$, i.e., whenever, $2\omega_m t = 2n\pi$

Or, whenever, $t = \frac{n\pi}{\omega_m} = \frac{n\pi}{\frac{\omega_2-\omega_1}{2}} = \frac{2n\pi}{\omega_2-\omega_1}$, here n = 1,2,3,……….etc.

Now, if the first maximum amplitude appears at time t_1, the second one at time t_2, the third one at time t_3, and so on, then

$t_1 = \frac{2\pi}{\omega_2-\omega_1}$, $t_2 = \frac{4\pi}{\omega_2-\omega_1}$, $t_1 = \frac{6\pi}{\omega_2-\omega_1}$, ……..and so on

And, time between two successive maxima would be, $\Delta t = t_2 - t_1 = t_3 - t_2 = \cdots\cdots = \frac{2\pi}{\omega_2-\omega_1}$

Hence, frequency of appearance of maxima, $\nu = \frac{2\pi}{\Delta t} = \frac{2\pi}{\frac{2\pi}{\omega_2-\omega_1}} = \omega_2 - \omega_1$

Condition for minimum amplitude:

Resultant amplitude assumes minimum value, $A(t) = (A_1 \sim A_2)$,

Whenever, $\cos(2\,\omega_m\,t) = -1$, i.e., whenever, $2\omega_m t = (2n+1)\pi$

Or, whenever, $t = \frac{(2n+1)\pi}{2\omega_m} = \frac{(2n+1)\pi}{2X\frac{\omega_2-\omega_1}{2}} = \frac{(2n+1)\pi}{\omega_2-\omega_1}$, here n = 0,1,2,3,……….etc.

Now, if the first minimum amplitude appears at time t_1, the second one at time t_2, the third one at time t_3, and so on, then

$t_1 = \frac{\pi}{\omega_2-\omega_1}$, $t_2 = \frac{3\pi}{\omega_2-\omega_1}$, $t_1 = \frac{5\pi}{\omega_2-\omega_1}$, ……..and so on

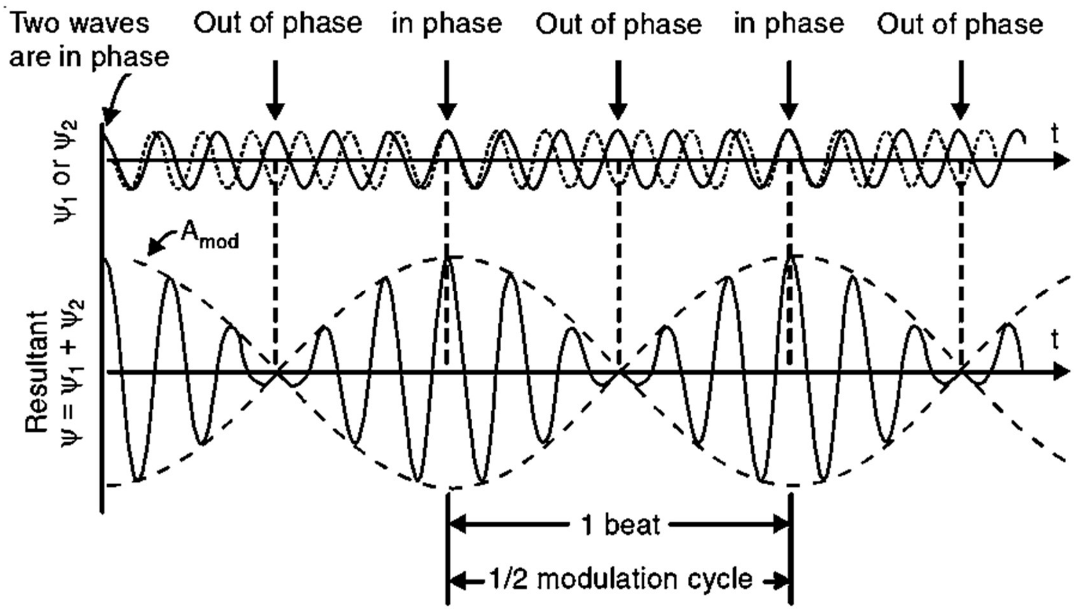
And, time between two successive minima would be, $\Delta t = t_2 - t_1 = t_3 - t_2 = \cdots\cdots = \frac{2\pi}{\omega_2-\omega_1}$

Hence, frequency of appearance of minima, $\nu = \frac{2\pi}{\Delta t} = \frac{2\pi}{\frac{2\pi}{\omega_2-\omega_1}} = \omega_2 - \omega_1$

This frequency is called beat frequency and this is audible if $(\omega_2 \sim \omega_1) < 10$ due to persistence of hearing.

Hence, the number of beats produced per second is equal to the reciprocal of time interval between two successive minima.

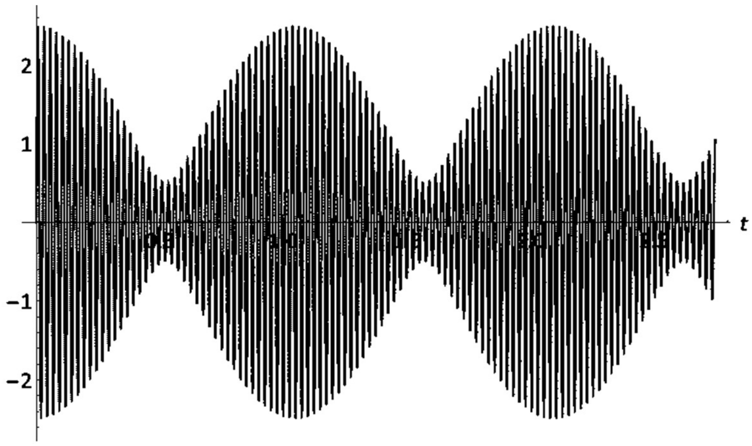
Graphical representation of appearance of beats:



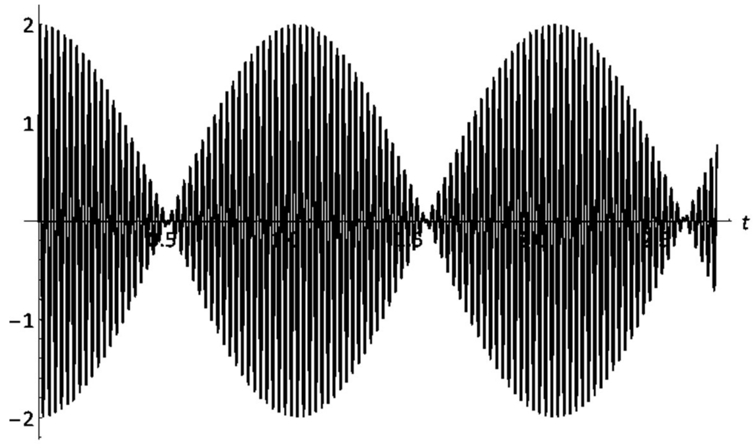
Hence, when two waves meet in phase the amplitude of resultant oscillation is maximum, i.e., $A(t) = (A_1 + A_2)$ and when two waves meet out of phase the amplitude of resultant oscillation is minimum, i.e., $A(t) = (A_1 \sim A_2)$

Note:

1. When amplitudes of superposing waves are not equal then amplitude of resultant oscillation at minima would not be zero and the waveform will appear as follows:



But when amplitudes of superposing waves is equal, amplitude of resultant oscillation at minima would be zero and the waveform will appear as follows:



And then minima and maxima would be more distinct

<u>Application of Beats</u>

The phenomenon of beats is of great importance. Beats can be used to determine the small difference between frequencies of two sources of sound. There are many physical phenomena which involve beats. For example,

a) Musicians often make use of beats in tuning their instruments. If the instrument is out of tune, one will hear beat deliberately produced in a particular section of an orchestra to give a pleasing tone to the resulting sound.

b) When two tuning forks of same frequency are sounded, a continuous sound is heard. When one of the tuning forks is waxed a little, so as to reduce its frequency, beats are heard because now the frequencies of the two tuning forks are slightly different. By counting the number of beats heard in a given interval of time, one can calculate the beat frequency.

c) Beats can be used to adjust the vibrating length between the two bridges in Sonometer Experiment.

d) Beats can be used in detection of harmful gases in mines. The presence of dangerous gases in mines is detected by use of an apparatus .The apparatus consists of two identical pipes; one blown from the air from reservoir and other by the air in the mine. If the air from mines as as pure as the air from reservoir, the two pipes will give the notes of same frequency. But if the air in the mines is polluted with dangerous gases, then due to presence of such gases, which usually are lighter than air would be having a greater velocity of sound waves and that pipe will give a note of higher frequency. And in that case sounds from two pipes will produce beats. Thus, the method serve as an early warning system to safeguard workers against possible dangerous explosions.

Numerical problems:

1. Two sitar strings A and B playing the note 'Dha' are slightly out of tune and produce beats of frequency 5 Hz. The tension of the string B is slightly increased and the beat frequency is found to decrease to 3 Hz. What is the original frequency of B if the frequency of A is 427 Hz?
2. Two radio stations broadcast their programmes at the same amplitude A, and at slightly different frequencies, where their difference is equal to signals from the two stations simultaneously. Find the time interval between of the intensity of the signal received by the detector.
3. Two tuning forks when sounded together give 4 beats per second. One is unison with a length of 96 cm of a sonometer wire under certain tension and the other with 96.8 cm of that wire under the same tension. Find the frequencies of these forks.
   [frequency of vibration (ν)of stretched wire of lenth *'l'*, tension *'T'* and mass per unit length 'μ' is given by'$\nu = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$]

In next class: In next class we shall discuss Superposition of N collinear Harmonic Oscillations with (1) equal phase differences and (2) equal frequency differences.